Inter (Part-II) 2019

Mathematics	Group-I	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) Define explicit function.

Ans If x and y are so mixed up and y cannot expressed in terms of the independent variable x, then y is called an implicit function. For example,

1. $x^2 + xy + y^2 = 0$
2. $\dfrac{xy^2 - y + 9}{xy} = +1$ are implicit functions of x and y.

Symbolically, it is written as $f(x, y) = 0$.

(ii) Determine whether the function $f(x) = x\sqrt{x^2+5}$ is even or odd.

Ans

$$f(x) = x\sqrt{x^2+5}$$
$$f(-x) = -x\sqrt{(-x)^2+5}$$
$$= -x\sqrt{x^2+5}$$
$$= -fx$$

$\therefore$ f(x) is odd.

(iii) Prove that $\lim\limits_{x\to 0}\dfrac{\sqrt{x+a}-\sqrt{a}}{x} = \dfrac{1}{2\sqrt{a}}$.

Ans By substituting x = 0, we have $\left(\dfrac{0}{0}\right)$ form, so rationalizing the numerator.

$$\therefore \lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x} = \lim_{x\to 0}\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\right)\left(\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$= \lim_{x\to 0}\frac{x+a-a}{x(\sqrt{x+a}+\sqrt{a})} = \lim_{x\to 0}\frac{x}{x(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+a}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

(iv) If $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$, find $\dfrac{dy}{dx}$.

Ans

$$y = \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= (x)^{1/2} - \frac{1}{(x)^{1/2}} = (x)^{1/2} - (x)^{-1/2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{1/2} - x^{-1/2})$$

$$= \frac{d}{dx}x^{1/2} - \frac{d}{dx}x^{-1/2} = \frac{1}{2}x^{-1/2} - \left(\frac{-1}{2}\right)(x)^{-3/2}$$

$$= \frac{1}{2x^{1/2}} - \left(\frac{-1}{2}\right)x^{-3/2} = \frac{1}{2x^{1/2}} + \frac{1}{2(x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{x+1}{2x^{3/2}}$$

(v) Find $\frac{dy}{dx}$, if $x^2 + y^2 = 4$.

Ans Here $x^2 + y^2 = 4$ (1)

Differentiating both sides of (1) w.r.t x., we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solving (i) for y in terms of x, we have

$$y = \pm\sqrt{4 - x^2}$$

$\Rightarrow$ $$y = \sqrt{4 - x^2} \quad (2)$$

or $\Rightarrow$ $$y = -\sqrt{4 - x^2} \quad (3)$$

$\frac{dy}{dx}$ found above represents the derivative of each of functions defined as in (2) and (3).

From (2), $$\frac{dy}{dx} = \frac{1}{2\sqrt{4 - x^2}} \times (-2x) = -\frac{-x}{\sqrt{4 - x^2}}$$

$$= -\frac{x}{y} \qquad (\because \sqrt{4 - x^2} = y)$$

From (iii) $$\frac{dy}{dx} = -\frac{1}{2\sqrt{4 - x^2}} \times (-2x)$$

$$= \frac{-x}{\sqrt{4 - x^2}} = -\frac{x}{y} \qquad (\because -\sqrt{4 - x} = y)$$

(vi) Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$.

Ans Let $y = \tan^{-1} x$ (1)

$x = \tan y$ or $x = \tan y$

for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (2)

Differentiating both sides of (2) w.r.t. 'x', we have

$$1 = \frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y)\frac{dy}{dx}$$

$$= \sec^2 y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} \quad \text{for} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$= \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2} \quad \text{for} \quad x \in R$$

Thus $\frac{dy}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$ for $x \in R$

(vii) Differentiate $\sin^{-1}\sqrt{1 - x^2}$ w.r.t. 'x'.

Ans Let $y = \sin^{-1}\sqrt{1 - x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\sqrt{1 - x^2}\right)$$

$$= \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \cdot \frac{d}{dx}(1 - x^2)^{1/2}$$

$$= \frac{1}{\sqrt{1 - (1 - x^2)}} \frac{1}{2}(1 - x^2)^{-1/2} (0 - 2x)$$

$$= \frac{-x}{\sqrt{1 - 1 + x^2} \cdot \sqrt{1 - x^2}} = -\frac{x}{x(\sqrt{1 - x^2})}$$

$$= -\frac{1}{\sqrt{1 - x^2}}$$

(viii) Differentiate $y = a^{\sqrt{x}}$.

Ans Let $u = \sqrt{x}$ Then $y = a^u$ (A)

and $\frac{du}{dx} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Differentiating both sides of (A) w.r.t. 'x' gives

$$\frac{dy}{dx} = \frac{d}{dx}(a^u) = \left(\frac{d}{du}\right) a^u \frac{du}{dx} \quad \left(\because \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\right)$$

$$= (a^u \ln a) \cdot \frac{du}{dx} \quad \left(\text{Using } \frac{d}{dx}(a^x) = a^x \ln a\right)$$

Thus $\frac{d}{dx}(a^{\sqrt{x}}) = (a^{\sqrt{x}} \ln a) \cdot \frac{1}{2\sqrt{x}}$

$$\left(\because \quad u = \sqrt{x} \quad \text{and} \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\ln a}{2} \cdot a^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}.$$

(ix) Prove that $\frac{d}{dx}(\cosh x) = \sin h\, x$.

Ans $\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right]$

$$= \frac{1}{2}[e^x + e^{-x} \cdot (-1)]$$

$$= \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

(x) Find $\frac{dy}{dx}$, if $y = (x + 1)^x$.

Ans $y = (x + 1)^x$

$\ln y = \ln (x + 1)^x$

$\ln y = x \cdot \ln (x + 1)$

Differentiate w.r.t. x.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[x \cdot \ln(x + 1)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \frac{d}{dx}[\ln(x + 1)] + \ln(x + 1) \frac{d}{dx}(x)$$

$$= x\left[\frac{1}{x + 1} \cdot (1 + 0)\right] + \ln (x + 1) \cdot 1$$

$$\frac{dy}{dx} = y\left[\frac{x}{x + 1} + \ln (x + 1)\right]$$

$$\therefore \quad \frac{dy}{dx} = (x + 1)^x\left[\frac{x}{x + 1} + \ln (x + 1)\right]$$

(xi) Define decreasing function. Give an example.

Ans Let f be defined on an interval (a, b) and let $x_1, x_2 \in (a, b)$, then f is decreasing function on the (a, b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

(xii) Determine $f(x) = \cos x$ is increasing or decreasing in the interval $\left(\frac{\pi}{2}, \pi\right)$.

Ans $f(x) = \cos x$

$f'(x) = -\sin x$

$f'(x) = -ve \ \forall \ x(\frac{\pi}{2}, \pi)$

$\therefore$ f(x) is decreasing.

3. Write short answers to any EIGHT (8) questions: (16)

(i) What is differential coefficient?

Ans Instead of dy, we can write df, that is, $df = f'(x)\,dx$ where $f'(x)$ being coefficient of differential is called differential coefficient.

(ii) Evaluate $\int \frac{e^{2x} + e^x}{e^x} dx$.

Ans

$$\int \frac{e^{2x} + e^x}{e^x} dx = \int (e^x + 1)\,dx$$

$$\left(\because \ \frac{e^{2x}}{e^x} = e^{2x-x} = e^x\right)$$

$$\int e^x\,dx + \int 1\,dx = e^x + x + c$$

(iii) Integrate by substitution $\int \frac{-2x}{\sqrt{4 - x^2}} dx$.

Ans If we put $u = 4 - x^2$, then

$du = -2x\,dx$

Thus $\int \frac{-x}{\sqrt{4 - x^2}} dx = \int (4 - x^2)^{-1/2} (-2x)\,dx = \int u^{-1/2}\,dx$

$$= \frac{(u)^{-1/2+1}}{\frac{-1}{2} + 1} + c = \frac{u^{1/2}}{\frac{1}{2}} + c$$

$$= 2\sqrt{u + c} = 2\sqrt{4 - x^2} + c$$

(iv) Find the integral $\int \frac{\cos x}{\sin x \ln(\sin x)} dx$.

Ans

$$\int \frac{1}{\ln \sin x}\left(\frac{\cos x}{\sin x}\right) dx,$$

If we put $u = \ln \sin x$.

Then $du = \frac{1}{\sin x} \cdot \cos x\,dx = \frac{\cos x}{\sin x} dx$

Thus $\int \frac{1}{\ln \sin x}\left(\frac{\cos x}{\sin x}\right) dx = \int \frac{1}{u} du = \ln u + c = \ln(\ln \sin x) + c$

(v) **Evaluate integral by parts $\int x . \sin x \, dx$.**

Ans $\int x \sin x \, dx \Rightarrow x(-\cos x) - \int (-\cos x) \frac{d}{dx}(x) \, dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c = \sin x - x \cos x + c$

(vi) **Find indefinite integral $\int e^{ax}\left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}}\right] dx$.**

Ans Let $\sec^{-1} x = f(x)$. Then $f'(x) = \frac{1}{x\sqrt{x^2-1}}$.

Thus $\int e^{ax}\left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}}\right] dx$

$\int e^{ax} (f(x) + f'(x)) \, dx$

$\int e^{ax} (a f(x) + f'(x)) \, dx$

$= \int \frac{d}{dx} (e^{ax} f(x)) \, dx$

$= e^{ax} . f(x) + c = e^{ax} \sec^{-1} x + c$

(vii) . **Evaluate $\int \frac{5x+8}{(x+3)(2x-1)} dx$ using partial fraction.**

Ans We write $\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

$5x + 8 = A(2x-1) + B(x+3)$ which gives

$2A + B = 5$ (i) (Equating coefficients of x)

$-A + 3B = 8$ (ii) (Equating constant terms)

$-2A + 6B = 16$ (iii) (Multiplying eq. (ii) by 2)

$7B = 21$ (By adding (i) and (iii))

$B = 3$

From eq. (ii) $A = 3B - 8 = 3(3) - 8 = 9 - 8 = 1$

Thus, $\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$

Now, $\int \frac{5x+8}{(x+3)(2x-1)} dx = \int \left[\frac{1}{x+3} + \frac{3}{2x-1}\right] dx$

$= \int \frac{1}{x+3} dx + \int \frac{3}{2x-1} dx = \int \frac{1}{x+3} dx + \frac{3}{2} \int \frac{1}{2x-1} . 2 \, dx$

$= \ln|x+3| + \frac{3}{2} \ln|2x-1| + c$

(viii) Define definite integral.

Ans The integral of f from a to b, is denoted by $\int_a^b f(x)\,dx$, and evaluated as

$$\int_a^b f(x)\,dx = \int_a^b \phi'(x)\,dx \qquad (\text{If } f(x) = \phi'(x))$$
$$= \phi(b) + c) - (\phi(a) + c)$$
$$= \phi(b) - \phi(a)$$

Since $\phi(b) - \phi(a)$ is definite number, that is,

$\int_a^b (fx)\,dx$ has a definite value, therefore, it is called the definite integral of f from a to b (a and b are lower and upper limits, respectively).

(ix) Calculate the integral $\int_0^{\pi/4} \sec x(\sec x + \tan x)\,dx$.

Ans $\int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx$

$$= \int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} \sec x \tan x\,dx$$
$$= [\tan x]_0^{\pi/4} + [\sec x]_0^{\pi/4} = \left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\sec\frac{\pi}{4} - \sec 0\right)$$
$$= (1 - 0) + (\sqrt{2} - 1) = \sqrt{2}$$

(x) If $\int_{-2}^{1} f(x)dx = 5$, $\int_{-2}^{1} g(x)\,dx = 4$, then evaluate $\int_{-2}^{1} [3f(x) - 2g(x)]\,dx$.

Ans $\int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2g(x)\,dx = 3\int_{-2}^{1} f(x)\,dx - 2\int_{-2}^{1} g(x)\,dx$

$$= (3 \times 5) - (2 \times 4) = 15 - 8 = 7$$

(xi) If a non-vertical line divides a plane into two, then write the name of that two planes.

Ans Non-vertical lines means horizontal. Line will divide into upper and lower half.

(xii) Graph the inequality $x + 3y > 6$.

Ans $x + 3y > 6$

Associated eq.,

$x + 3y = 6$

x	−3	0	3
y	3	2	1

Check point (0, 0)

$$x + 3y > 6$$
$$0 + 3(0) > 6$$
$$0 + 0 > 6$$

False (not true) (0, 0) is not the part of the solution.

4. Write short answers to any NINE (9) questions: (18)

(i) Find coordinates of the point that divide the join of A(–6, 3) and B(5, –2) in the ratio 2 : 3 internally.

Ans Here $k_1 = 2, k_2 = 3, x_1 = -6, x_2 = 5.$

$y_1 = 3, y_2 = -2$

By the formula, we have

$$x = \frac{k_1x_2 - k_2x_1}{k_1 + k_2}\ y = \frac{k_1y_2 - k_2y_1}{k_1 + k_2}$$

$$x = \frac{2 \times 5 + 3 \times (-6)}{2 + 3} = \frac{10 - 18}{5} = \frac{-8}{5}$$

and $y = \dfrac{2(-2) + 3(3)}{2 + 3} = \dfrac{-4 + 9}{5} = \dfrac{5}{5} = 1.$

So, coordinates of the required point are $\left(\frac{-8}{5}, 1\right)$.

(ii) Show that the triangle with vertices A(1, 1), B(4, 5) and C(12, –5) is right triangle.

Ans Slope of AB $= m_1 = \dfrac{5 - 1}{4 - 1} = \dfrac{4}{3}$

and Slope of BC $= m_2 = \dfrac{-5 - 5}{12 - 4} = \dfrac{-10}{8} = \dfrac{-5}{4}$

Slope of AC $= m_3 = \dfrac{-5 - 1}{12 - 1} = \dfrac{-6}{11}$

Since $m_1 . m_2 = \left(\frac{4}{3} \times \frac{-5}{4}\right) = \frac{-5}{3} \neq -1.$

Therefore, ΔABC is not a right triangle.

(iii) Find an equation of the line through (–4, –6) and perpendicular to the line having slope $-\frac{3}{2}$.

Ans The slope of required line $= -\left(-\dfrac{2}{3}\right)$

$$= \frac{2}{3}$$

Thus the equation of required line is

$$y-(-6)=\frac{2}{3}(x-(-4))$$

$$3(y+6)=2(x+4)$$

$$3y+18=2x+8$$

$$\Rightarrow \quad 2x-3y-10=0$$

(iv) Define trapezium.

Ans A quadrilateral having two parallel sides and two non-parallel sides. ABCD is a trapezium.

(v) Define parabola.

Ans The set of all points in a plane equidistance from a fixed line *l* and a fixed point F not on the line *l* is called a parabola. The fixed line *l* is called a directrix and the fixed point f is called a focus of the parabola.

(vi) Check the position of the point (5, 6) with respect to the circle $2x^2+2y^2+12x-8y+1=0$.

Ans

$$2x^2+2y^2+12x-8y+1=0$$

$$2\left(x^2+y^2+6x-4y+\frac{1}{2}\right)=0$$

$$\therefore \quad x^2+y^2+6x-4y+\frac{1}{2}=0$$

$$x^2+y^2+2(3)x+2(-2)y+\frac{1}{2}=0$$

Putting (5, 6) in it,

$$\text{L.H.S}=(5)^2+(6)^2+2(3)(5)+2(-2)(6)+\frac{1}{2}$$

$$=25+36+30-24+\frac{1}{2}=67+\frac{1}{2}=67.5>0$$

Hence (5, 6) lies outside the given circle.

(vii) Find eccentricity of the ellipse $x^2+4y^2=16$.

Ans

$$x^2+4y^2=16$$

Dividing both sides by 16

$$\frac{x^2}{16}+\frac{4y^2}{16}=\frac{16}{16}$$

$$\frac{x^2}{16}+\frac{y^2}{4}=1 \qquad \text{(i)}$$

Comparing eq. (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b), we have

$$a^2 = 16 \qquad b^2 = 4$$

$$\Rightarrow \quad a = \pm 4 \qquad b = \pm 2$$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4} \Rightarrow e = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

(viii) Find an equation of hyperbola if its foci (0, ± 9) and directrices y = ± 4.

Ans $f(0, 9), f'(0, -9)$

$$ae = 9, \frac{a}{e} = 4 \quad \Rightarrow \quad a = 4e$$

$$4e\,(e) = 9$$

$$4e^2 = 9 \quad \Rightarrow \quad e^2 = \frac{9}{4} \quad \Rightarrow \quad e = \frac{3}{2}$$

$$a = 4e \quad \Rightarrow \quad 4\left(\frac{3}{2}\right) = 6$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{(6)^2 + b^2}{(6)^2} = \frac{36 + b^2}{36} = \frac{9}{4}$$

$$36 + b^2 = \frac{9}{4}(36)$$

$$36 + b^2 = 81$$

$$b^2 = 81 - 36 = 45$$

Equation of required hyperbola is

$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \Rightarrow \quad -\frac{x^2}{45} + \frac{y^2}{36} = 1$$

(ix) If $\overrightarrow{AB} = \overrightarrow{CD}$, find coordinates of point A. If B, C, D are (1, 2), (–2, 5), (4, 11).

Ans Let A(x, y), B(1, 2), C(–2, 5), D(4, 11)

$$\overrightarrow{AB} \parallel \overrightarrow{CD} \qquad \text{(i)}$$

$$\overrightarrow{AB} = (1 - x)\,\underline{i} + (2 - y)\,\underline{j} \qquad \text{(ii)}$$

$$\overrightarrow{CD} = (4 + 2)\,\underline{i} + (11 - 5)\,\underline{j} = 6\underline{i} + 6\underline{j} \qquad \text{(iii)}$$

Putting (ii) and (iii) in (i),

$\therefore \quad 1 - x = 6 \qquad 2 - y = 6$

$\because \quad -x = 5 \qquad \therefore \quad -y = 4$

$\therefore \quad x = -5 \qquad \therefore \quad y = -4$

$\therefore \quad A(x, y) = (-5, -4)$

(x) Write direction cosine of $\overrightarrow{PQ}$, if P(2, 1, 5), Q(1, 3, 1).

Ans P(2, 1, 5), Q(1, 3, 1)

$$\overrightarrow{PQ} = (1-2)\underline{i} + (3-1)\underline{j} + (1-5)\underline{k}$$
$$= -\underline{i} + 2\underline{j} - 4\underline{k}$$
$$|\overrightarrow{PQ}| = \sqrt{(1)^2 + (2)^2 + (-4)^2}\ \sqrt{1 + 4 + 16} = \sqrt{21}$$

$\therefore$ Direction cosines of $\overrightarrow{PQ}$ are

$$\left(\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right)$$

(xi) Show that vectors $3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} + 5\underline{k}$ and $2\underline{i} + \underline{j} - 4\underline{k}$ form a right triangle.

Ans Let
$$\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$$
$$\underline{b} = \underline{i} - 3\underline{j} + 5\underline{k}$$
$$\underline{c} = 2\underline{i} + \underline{j} - 4\underline{k}$$

Since, $\underline{b} + \underline{c} = (\underline{i} - 3\underline{j} + 5\underline{k}) + (2\underline{i} + \underline{j} - 4\underline{k})$
$$= 3\underline{i} - 2\underline{j} + \underline{k}$$
$$= \underline{a}$$

Therefore, $\underline{a}$, $\underline{b}$ and $\underline{c}$ are the sides of a triangle.

Since $\underline{a} \cdot \underline{c} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (2\underline{i} + \underline{j} - 4\underline{k})$
$$= (3 \times 2) + (-2 \times 1) + (1 \times -4)$$
$$= 6 - 2 - 4 = 0$$

Therefore, $\underline{a}$, $\underline{b}$ and $\underline{c}$ are the sides of a right triangle.

(xii) Find unit vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$, if $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$.

Ans $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$
$$= \underline{i}(-4 - 3) - \underline{j}(-4 + 2) + \underline{k}(3 + 2)$$
$$= -7\underline{i} + 2\underline{j} + 5\underline{k}$$
$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2}$$
$$= \sqrt{49 + 4 + 25} = \sqrt{78}$$

Required unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$= \frac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$$

$$= \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$$

(xiii) A force $\underline{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at P(1, –2, 3). Find its moment about the point Q(2, 1, 1).

Ans $\overrightarrow{QP} = (1-2)\underline{i} + (-2-1)\underline{j} + (3-1)\underline{k} = -\underline{i} - 3\underline{j} + 2\underline{k}$

Moment of $\vec{F}$ about Q = $\overrightarrow{QP} \times \vec{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

$$= (9-8)\underline{i} - (3-14)\underline{j} + (-4+21)\underline{k}$$

$$= \underline{i} + 11\underline{j} + 17\underline{k}$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Find the values of 'm' and 'n' so that **(5)**

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases} \text{ is continuous at } x = 3.$$

Ans

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases}$$

L.H. lim = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} mx = 3m$

R.H. lim = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-2x + 9) = -6 + 9 = 3$

Since f(x) is continuous $\therefore$ L.H. lim = R.H. lim

$3m = 3$

$\boxed{m = 1}$

Also $f(3) = n$ Also $\lim_{x \to 3^+} f(x) = 3$

Since $\lim_{x \to 3^+} f(x) = f(3)$

$\boxed{3 = n}$

(b) If $y = e^x \cdot \sin x$, then prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. (5)

Ans

$$y = e^x \sin x \quad (1)$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x)$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + e^x \cdot \sin x \quad (2)$$

$$= e^x (\sin x + \cos x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x(\sin x + \cos x)]$$

$$= e^x \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}(e^x)$$

$$= e^x (\cos x - \sin x) + e^x (\sin x + \cos x)$$

$$= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x$$

$$= 2e^x \cos x \quad (3)$$

From eq. (2),

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$= e^x \cos x + y \qquad \text{[From (1)]}$$

$$\text{or } e^x \cos x = \frac{dy}{dx} - y \quad (4)$$

Putting the value of eq. (4) in eq. (3),

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - y\right)$$

$$= 2\frac{dy}{dx} - 2y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \qquad \text{Proved.}$$

Q.6.(a) Evaluate $\int \frac{\sqrt{2}}{\sin x + \cos x} dx$. (5)

Ans

$$= \int \frac{dx}{\frac{\sin x + \cos x}{\sqrt{2}}}$$

$$= \int \frac{dx}{\cos x \cdot \frac{1}{\sqrt{2}} + \sin x \cdot \frac{1}{\sqrt{2}}} \quad (1)$$

$$= \int \frac{dx}{\cos x \cdot \cos\frac{\pi}{4} + \sin x \cdot \sin\frac{\pi}{4}}$$

$$= \int \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)} = \int \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \ln\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right| + C$$

(b) Find an equation of the perpendicular bisector of the segment joining the points A(3, 5) and B(9, 8). (5)

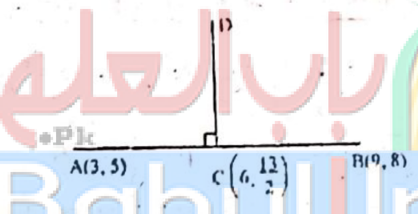
Ans Let C be the mid-point of A and B. Its coordinates are:

$$C\left(\frac{3+9}{2}, \frac{5+8}{2}\right) = C\left(6, \frac{13}{2}\right)$$

$$\text{Slope of AB} = \frac{8-5}{9-3} = \frac{3}{6} = \frac{1}{2}$$

$$\text{Slope of CD} = m = -\frac{2}{1} = -2$$

Eq. of perpendicular bisector CD through $\left(6, \frac{13}{2}\right)$ is:



$$y - y_1 = m(x - x_1)$$

$$y - \frac{13}{2} = -2(x - 6)$$

$$= -2x + 12$$

$$2x + y - \frac{13}{2} - 12 = 0$$

$$2x + y - \frac{37}{2} = 0$$

$$4x + 2y - 37 = 0$$

Q.7.(a) Solve the differential equation $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$. (5)

Ans

$$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$$

$$x^2(1-y)\frac{dy}{dx} + y^2(1+x) = 0$$

$$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$-\left(\frac{1+y}{y^2}\right)dy = \left(\frac{1+x}{x^2}\right)dx$$

$$\left(\frac{-1}{y^2} + \frac{y}{y^2}\right)dy = \left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$$

$$-y^{-2}dy + \frac{dy}{y} = x^{-2}\,dx + \frac{dx}{x}$$

$$-\int y^{-2}\,dy + \int\frac{dy}{y} = \int x^{-2}\,dx + \int\frac{dx}{x} + C$$

$$\frac{-y^{-1}}{-1} + \ln|y| = \frac{x^{-1}}{-1} + \ln|x| + C$$

$$\frac{1}{y} + \ln|y| = -\frac{1}{x} + \ln|x| + C$$

(b) Graph the solution region of the following system of linear inequalities and find the corner points: (5)

$x + y \le 5$, $-2x + y \le 2$, $y \ge 0$

Ans

$$x + y \le 5\ ,\ -2x + y \le 2\ ,\ y \ge 0$$

$x + y \le 5$ (i)

$x + y = 5$ (iii)

Putting $x = 0$ in eq. (iii),

$0 + y = 5 \Rightarrow y = 5$

$\therefore$ (0, 5) is a point on (iii)

Putting $y = 0$ in eq. (iii),

$x + 0 = 5 \Rightarrow x = 5$

$\therefore$ (5, 0) is another point on (iii)

Putting $x = 0, y = 0$ in eq. (i),

$0 + 0 < 5$

$0 < 5$

Which is true. Hence solution region of (i) lies on origin-side of (i).

$-2x + y \le 2$ (ii)

$-2x + y = 2$ (iv)

Putting $x = 0$ in eq. (iv),

$0 + y = 2 \Rightarrow y = 2$

$\therefore$ (0, 2) is a point on (iv)

Putting $y = 0$ in eq. (iv),

$-2x + 0 = 2 \Rightarrow x = -1$

$\therefore$ (–1, 0) is another point on (iv)

Putting $x = 0, y = 0$ in eq. (ii),

$-0 + 0 < 2$

$0 < 2$

Which is true. Hence solution region of (ii) lies on origin-side of (ii).

Q.8.(a) Find the lines represented by each of the following and also find measure of the angle between them $x^2 + 2xy\sec\alpha + y^2 = 0$. (5)

Ans

$$x^2 + 2xy\sec\alpha + y^2 = 0 \quad \text{(i)}$$

The given equation can be written as

$$x^2 + 2xy \sec\alpha + y^2 = 0$$
$$y^2 + 2xy \sec\alpha + x^2 = 0$$
$$\frac{y^2}{x^2} + \frac{2xy \sec\alpha}{x^2} + \frac{x^2}{x^2} = 0$$
$$\left(\frac{y}{x}\right)^2 + 2\sec\alpha\left(\frac{y}{x}\right) + 1 = 0$$

which is the quadratic equation in $\frac{y}{x}$.

$$\frac{y}{x} = \frac{-2\sec\alpha \pm \sqrt{(2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{-2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2}$$
$$= \frac{-2\sec\alpha \pm 2\sqrt{\tan^2\alpha}}{2}$$
$$= -\sec\alpha \pm \tan\alpha$$

$$\Rightarrow \quad \frac{y}{x} = -\frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\alpha} = \frac{-1 \mp \sin\alpha}{\cos\alpha}$$

$\Rightarrow \quad \cos\alpha\, y = -(1 - \sin\alpha)\, x \quad$ or $\quad \cos\alpha\, y = -(1 + \sin\alpha)$

Thus $\quad (1 - \sin\alpha)\, x + \cos\alpha\, y = 0 \quad$ and

$\quad (1 + \sin\alpha)\, x + \cos\alpha\, y = 0$

are the required lines.

Let θ be the measure of the angle between the lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$
$$= \frac{2\sqrt{\sec^2\alpha - 1}}{1 + 1} = \sqrt{\tan^2\alpha} = \tan\alpha$$

$\Rightarrow \quad \theta = \alpha \qquad (\because \quad a = 1,\ h = \sec\alpha,\ b = 1)$

(b) Find the coordinates of the points of intersection of the line $2x + y + 5 = 0$ and the circle $x^2 + y^2 + 2x - 9 = 0$. Also find the length of intercepted chord. (5)

Ans From $\quad 2x + y = 5$, we have

$\quad y = (5 - 2x)$

Inserting this value of y into equation of the circle, we get

$$x^2 + (5 - 2x)^2 + 2x - 9 = 0$$
$$x^2 + (25 - 20x + 4x^2) + 2x - 9 = 0$$

$$x^2 + 25 - 20x + 4x^2 + 2x - 9 = 0$$
$$5x^2 - 18x + 16 = 0$$

By applying quadratic formula,

$$\Rightarrow \quad x = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(5)(16)}}{2(5)}$$
$$= \frac{18 \pm \sqrt{324 - 320}}{10} = \frac{18 \pm \sqrt{4}}{10} = \frac{18 \pm 2}{10}$$
$$= \frac{20}{10}, \frac{16}{10} = \left(2, \frac{8}{5}\right)$$

When $x = 2$, $y = 5 - 4 = 1$

When $x = \frac{8}{5}$, $y = 5 - \left(\frac{16}{5}\right) = \frac{9}{5}$

Thus the points of intersection are P(2, 1) and Q $\left(\frac{8}{5}, \frac{9}{5}\right)$.

Length of the chord intercepted $= |\overline{PQ}| = \sqrt{\left(\frac{8}{5} - 2\right)^2 + \left(\frac{9}{5} - 1\right)^2}$

$$= \sqrt{\frac{4}{25} + \frac{16}{25}} = \sqrt{\frac{20}{25}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

Q.9.(a) Find equation of parabola with elements directrix : $x = -2$, focus (2, 2). **(5)**

Ans Directrix: $x = -2$ or $x + 2 = 0$

F(2, 2), P(x, y) from the graph, coordinates of vertex are: V(0, 2)

By the definition of parabola:

$$\left|\frac{PF}{PM}\right| = e = 1 \quad \Rightarrow \quad |PF| = |PM| \qquad \text{(i)}$$
$$|PF| = \sqrt{(x-2)^2 + (y-2)^2} \qquad \text{(ii)}$$
$$|PM| = \frac{|(1)(x) + (0)(y) + 2|}{\sqrt{(1)^2 + (0)^2}} = |x + 2| \qquad \text{(iii)}$$

Putting the value of eq. (ii) and eq. (iii) in eq. (i),

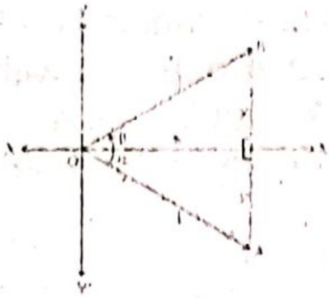
$$\sqrt{(x-2)^2 + (y-2)^2} = |x + 2|$$
$$\therefore \quad x^2 - 4x + 4 + y^2 - 4y + 4 = x^2 + 4x + 4$$
$$y^2 - 4y - 8x + 4 = 0$$

(b) Prove that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ by method of vectors. **(5)**

Ans

$$\cos\alpha = \frac{x}{1} \Rightarrow x = \cos\alpha$$

$$\sin\alpha = \frac{-y}{1} \Rightarrow y = -\sin\alpha$$



$\therefore$ $A(\cos\alpha, \sin\alpha)$

$$\cos\beta = \frac{x'}{1} = x' = \cos\beta$$

$$\sin\beta = \frac{y'}{1} = y' = \sin\beta$$

$\therefore$ $B(\cos\beta, \sin\beta)$

$$\overrightarrow{OA} = (\cos\alpha - 0)\underline{i} + (\sin\alpha - 0)\underline{j} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$$

$$\overrightarrow{OB} = (\cos\beta - 0)\underline{i} + (\sin\beta - 0)\underline{j} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}$$

$$\overrightarrow{OB} \times \overrightarrow{OA} = |\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha - \beta)\,\underline{k}$$
$$= (1)(1)\sin(\alpha + \beta)\,\underline{k}$$
$$= \sin(\alpha + \beta)\,\underline{k} \qquad \text{(i)}$$

Also $\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$

$$= \underline{i}(0 - 0) - \underline{j}(0 - 0) + \underline{k}(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$
$$= (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\,\underline{k} \qquad \text{(ii)}$$

From (i) and (ii),

$\sin(\alpha - \beta)\,\underline{k} = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\,k$

$\therefore$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$